Optimal Design for Accelerated-Stress Acceptance Test Based on Wiener Process

Chih-Chun Tsai, Chien-Tai Lin, and Narayanaswamy Balakrishnan

***Abstract*—Acceptance testing is widely used to assess whether a product meets the expectations of customers. Yet, traditional acceptance tests based on time-to-failure data will not be practical because today's highly reliable products may take a long time to fail. It may be good in this case to base a test on a suitable quality characteristic (QC) whose degradation over time is related to the reliability of the product. Motivated by resistor data, we first propose a degradation model to describe the degradation paths of the resistors. Next, we present an accelerated-stress acceptance test to reduce the acceptance testing time, and then derive the optimal accelerated-stress acceptance testing time for a product, and the probability of acceptance of the batch. A model incorporating cost is also used to determine the optimal design for an accelerated-stress acceptance experiment, and a motivating example is then presented to illustrate the proposed procedure. Finally, we examine the performance of the estimators, and the effect of misspecification of the parameters on the optimal test plan through a Monte Carlo simulation study, and a detailed sensitivity analysis.**

***Index Terms*—Cost function, optimal accelerated-stress acceptance testing time, optimal test plan, parameter misspecification, quality characteristic, sensitivity analysis.**

ACRONYMS AND ABBREVIATIONS

QC	quality characteristic
MLE	maximum likelihood estimate
BAN	best asymptotically normal
CI	confidence interval

NOTATION

$R_h(t_j)$	measured resistance of the h-th tested unit at time t_j
$R^{(k)}(t)$	measured resistance at time t under stress S_k for $k = 0, 1$
σ_k	diffusion coefficient for $k = 0, 1$
$L(t\|S_k)$	relative changes in the resistance of the resistors under stress S_k at time t
$B(\cdot)$	standard Brownian motion
t_{as}^*	optimal accelerated-stress acceptance testing time
TC	total cost of conducting an accelerated-stress acceptance test
C_{op}	unit cost of operation
C_{mea}	unit cost of measurement
C_{it}	unit cost of tested coupon
AVar	asymptotic variance
$\boldsymbol{\theta}$	vector of unknown model parameters
$\boldsymbol{\xi}$	test plan
$\boldsymbol{\xi}^*$	optimal test plan
$\mathbf{I}(\boldsymbol{\theta})$	Fisher information matrix

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I. INTRODUCTION

ACCEPTANCE testing is an important inspection procedure in reliability engineering and manufacturing. It is usually conducted for determining whether some pertinent characteristic of the unit meets the specification requirements of customers. Traditionally, this type of testing, including chemical tests, physical tests, or performance tests, has been done by recording the time of failure of each unit (see Gates & Fearey [4], Whitney [32], Mogg [13], Blumenthal [2], Vangel [27], and Ma & Robinson [10]). However, as modern production technology becomes increasingly sophisticated, manufactured products such as circuit boards, semiconductors, automobiles, and aerospace products are created with high quality and reliability; consequently, it takes a long time to monitor failures and to conduct inspection procedures for decision making. Usually, it is neither practical nor feasible to carry out such a test. In this situation, a suitable alternative is to base the method on data collected from degradation tests, which are especially useful in scenarios wherein there is a quality characteristic (QC) whose degradation over time is closely associated with the lifetime of the production.

The use of data from degradation tests has been studied extensively as an alternative to traditional life tests by a number of authors including Nelson [14], Meeker & Escobar [12], Bagdonavicius & Nikulin [1], Lawless [6], Gebraeel & Pan [5], Peng & Tseng [17], Yang [34], [36], Wang & Pham [29], and Peng *et al.* [18]. These references provide many examples from a wide range of applied areas. Most degradation data analysis and practices are based on the assumption of a Wiener process for the degradation paths of the products. Park & Padgett [15], and Liao & Elsayed [8] studied the use of accelerated degradation data in modelling and prediction through a Wiener process. Tang & Su

[21] discussed the estimation of the mean lifetime based on the intermediate degradation data from a Wiener process. Peng & Tseng [16] investigated the misspecification analysis of linear degradation models by incorporating the random effects in the drift coefficient, and measurement errors in a Wiener process. Recent developments and applications of Wiener degradation processes include those by Wang *et al.* [28], Tseng *et al.* [25], Si *et al.* [20], Tsai *et al.* [22], Wei *et al.* [30], and Ye *et al.* [38]. However, the design of an acceptance test in this framework has not been addressed yet.

A good degradation test plan based on an optimal choice of total sample size, inspection frequency, and total number of measurements can be used to achieve a specified degree of accuracy required for an efficient estimation of the model parameters. Boulanger & Escobar [3] described a method for planning accelerated degradation tests based on a nonlinear model that incorporates a sigmoidal growth curve. Yu & Tseng [41] proposed a quasi-linear model to address the associated optimal degradation design. Wu & Chang [33] utilized a nonlinear mixed integer programming technique to obtain an optimal degradation design. Marseguerra *et al.* [11] illustrated the use of multi-objective genetic algorithms for the design of optimal degradation tests. Some relevant literature and other approaches for optimal degradation designs can be found in the works of Yu & Chiao [40], Yu [39], Tseng *et al.* [26], Rathod *et al.* [19], Tsai *et al.* [23], [24], Ye *et al.* [37], and Yang [35]; but, until now, optimal degradation designs for acceptance tests have not been developed in the literature.

In this article, motivated by resistor data, a degradation model is proposed to describe the degradation paths of the resistors. Next, to shorten the duration of an acceptance test based on cost and time considerations, an accelerated-stress acceptance test is proposed. Then, an intuitive rule is used to determine the optimal accelerated-stress acceptance testing time, and the probability of acceptance of the batch is derived. We discuss the optimal design problem for the accelerated-stress acceptance experiment based on a model that incorporates the cost. Finally, a sensitivity analysis is conducted for examining the effects of misspecification of the parameters on the optimal test plan, and a simulation study is carried out for assessing the performance of the estimators of the model parameters as well.

The rest of this paper is organized as follows. Section II gives a motivating example, and Section III describes the formulation of the problem. Section IV presents the derivation of the optimal accelerated-stress acceptance testing time and the probability of acceptance of the batch, and discusses the corresponding optimal design based on a cost model. Section V uses the motivating example for illustrating the proposed method. The performance of the estimators of the model parameters, and the effect of misspecification of the parameters on the optimal test plan, are evaluated through a Monte Carlo simulation study, and a sensitivity analysis. Finally, some concluding remarks are made in Section VI, and all the technical details are relegated to the Appendix.

II. Motivating Example

The motivation for this research is a real example provided by a manufacturer of chip resistors in Taiwan. Chip resistors are passive components which are commonly used to create and maintain a safe level of current for many electrical products such as tablets, automobiles, and cell phones. When the resistance of a chip resistor deviates seriously away from its initial value, the performance of the device may be reduced significantly, and may even result in its breakdown. Fig. 1 displays the relative changes in the resistance of 60 resistors at 39 kΩ in a 1000-hour acceptance test at a temperature of 70°C.

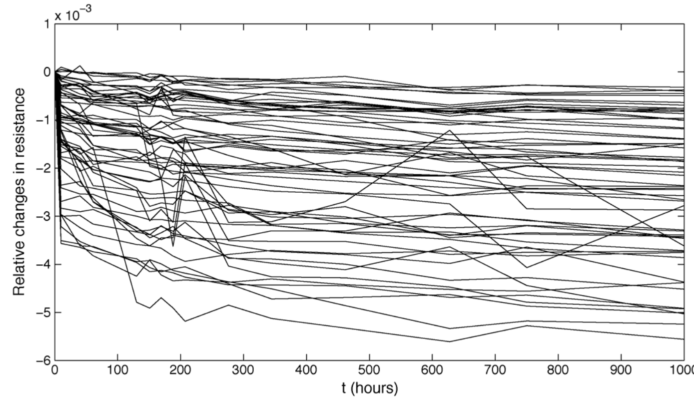

Fig. 1. Relative changes in the resistance of 60 resistors within 1000 hours.

Let n_0, and l_0 denote the total number of test units, and measurements under a prefixed temperature or stress $S_0 = 70^\circ\mathrm{C}$, respectively. Also, let $R_h(t_j)$ be the measured resistance of the h-th tested unit at time t_j, where $h = 1, \ldots, n_0$, and $j = 1, \ldots, l_0$. The commonly used policy demands that the batches of resistors are accepted only if the relative changes in resistance of the batch through the whole testing process under S_0 are all less than 1%; that is,

$$\left| \frac{R_h(t_j) - R_h(0)}{R_h(0)} \right| < 1\% \text{ for } 1 \le h \le n_0 \text{ and } 1 \le j \le l_0.$$

However, the use of such an acceptance testing time (termination time being $t_{l_0} = 1000$ hours) may be inefficient, or quite expensive in terms of time and cost, or both. Therefore, three practical decision problems arise naturally while developing an efficient acceptance test for the resistors.

1) To shorten the acceptance testing time, one may consider conducting an acceptance experiment at a higher temperature (say, $S_1 = 125^\circ\mathrm{C}$) to accelerate the chemical degradation process. How can one determine the optimal accelerated-stress acceptance testing time (denoted by t^*_{as}) in this accelerated test such that the performance of the relative changes in resistance under the new accelerated stress level S_1 will be close to the one under the original stress S_0 at time t_{l_0}, as illustrated in Fig. 2?
2) In some applications, it is very important to evaluate the performance of the products based on the probability of acceptance of the batch at time t^*_{as} while conducting an accelerated-stress acceptance experiment. For resistor data, how do we find the probability of acceptance?
3) How do we carry out an optimal accelerated-stress acceptance test? In other words, how many resistors should be taken for conducting an accelerated-stress acceptance test? How do we determine an appropriate inspection frequency? How many measurements should be made in one inspection to collect the desired degradation data?

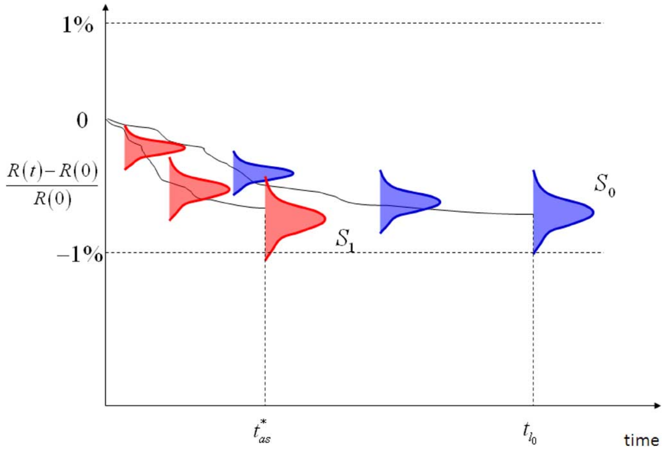

Fig. 2. Illustration of the optimal accelerated-stress acceptance testing time (t^*_{as}).

In the following section, we will formulate such an optimal design problem for an accelerated-stress acceptance test.

III. Formulation of the Problem

The plot in Fig. 1 shows that the degradation paths of the relative changes in the resistance of the resistors are exponentially decreasing over time t. Hence, the relative changes in the resistance of the resistors under stress S_k at time t can be suitably defined as

$$L(t|S_k)=\frac{R^{(k)}(t)-R^{(k)}(0)}{R^{(k)}(0)}=e^{-\alpha_k t^{\beta_k}+\sigma_k B(t^{\beta_k})}-1, \quad t>0. \tag{1}$$

Using a natural-logarithm transformation, we have

$$W_k(t)=\ln\left(L(t|S_k)+1\right)=-\alpha_k\tau_k(t)+\sigma_k B\left(\tau_k(t)\right), \quad t>0, \tag{2}$$

where $\tau_k(t) = t^{\beta_k}$.

As shown in Fig. 2, we want to determine the optimal accelerated-stress acceptance testing time t^*_{as} in the accelerated test S_1 such that the relative changes in resistance under accelerated stress S_1 are as close to the ones under the original stress S_0 at time t_{l_0} as possible. In general, if we wish to find an approximation of a distribution, we always look for tractable distributions whose means and standard deviations are the same as the target distribution, and then choose the one that performs best in the required criterion. Thus, a natural case in our situation would be to compare the sum of the absolute difference of the mean and standard deviation between the degradation paths $W_1(t)$ and $W_0(t_{l_0})$ for all t, and find its minimum. However, in most real-life problems, the computation of this absolute-difference optimization may not be a trivial issue. Hence, the use of the squared difference is proposed here; that is, the optimal accelerated-stress acceptance testing time t^*_{as} can be intuitively given by

$$t^*_{as} = \arg\min_t \Big\{\left[\mathrm{E}\left(W_1(t)\right)-\mathrm{E}\left(W_0\left(t_{l_0}\right)\right)\right]^2 + \left[\mathrm{Std}\left(W_1(t)\right)-\mathrm{Std}\left(W_0\left(t_{l_0}\right)\right)\right]^2\Big\}, \tag{3}$$

where $\mathrm{E}(W_k(t))$, and $\mathrm{Std}(W_k(t))$, respectively, are the mean, and standard deviation of the degradation path $W_k(t)$ at time t under stress S_k for $k = 0, 1$.

It is seen that t^*_{as} is a function of the unknown model parameters $\boldsymbol{\theta} = (\alpha_0, \beta_0, \sigma_0, \alpha_1, \beta_1, \sigma_1)$. In practice, these parameters will all be unknown. Hence, to estimate t^*_{as} efficiently, we propose to modify the earlier work on degradation test plans (see, for instance, Tsai et al. [24]) to conduct an accelerated-stress acceptance experiment for estimating t^*_{as} within the specified budgetary constraints as follows.

Suppose n_1 units are randomly selected for conducting an accelerated-stress acceptance experiment, and the measurements of each unit are made every f_1 units of time until time $t_{l_1} = f_1 l_1 t_{u_1}$, where l_1 is the number of measurements made under stress S_1, and t_{u_1} is one unit of time. It is clear that the values of the variables (n_1, f_1, l_1) will affect the experimental cost as well as the precision of the estimate of the optimal accelerated-stress acceptance testing time. Let $\mathrm{TC}(n_1, f_1, l_1)$ denote the total cost of conducting an accelerated-stress acceptance experiment, and $\hat{t}^*_{as}$ be the estimate of the optimal accelerated-stress acceptance testing time based on a test plan (n_1, f_1, l_1). Then, the typical decision problem of interest can be formulated as follows.

Minimize

$$\mathrm{AVar}\left(\hat{t}^*_{as}|n_1, f_1, l_1\right),$$

subject to

$$\mathrm{TC}(n_1, f_1, l_1) \le C_b, \quad n_1, f_1, l_1 \in \mathbb{N}^3, \tag{4}$$

where $\mathrm{AVar}(\hat{t}^*_{as})$ denotes the asymptotic variance of $\hat{t}^*_{as}$, and C_b is the total pre-fixed budget for conducting the accelerated-stress acceptance experiment.

In the next section, we discuss the solution to the above stated optimization problem.

IV. The Optimal Test Plan

To conduct an accelerated-stress acceptance test efficiently, the procedure for solving the optimization problem in (4) consists of three parts:

1) the derivation of the optimal accelerated-stress acceptance testing time,
2) the computation of the asymptotic variance of the estimate of the optimal accelerated-stress acceptance testing time, and
3) the total cost of the accelerated-stress acceptance experiment.

A. Expression of t^*_{as}

From (2), see that

$$\mathrm{E}\left(W_k(t)\right) = -\alpha_k\tau_k(t), \tag{5}$$

and

$$\mathrm{Std}\left(W_k(t)\right) = \sigma_k\sqrt{\tau_k(t)}. \tag{6}$$

Upon substituting (5) and (6) into (3), the optimal accelerated-stress acceptance testing time t^*_{as} can be obtained by minimizing the function

$$f(t) = \left[\alpha_1\tau_1(t) - \alpha_0\tau_0\left(t_{l_0}\right)\right]^2 + \left[\sigma_1\sqrt{\tau_1(t)} - \sigma_0\sqrt{\tau_0\left(t_{l_0}\right)}\right]^2. \tag{7}$$

The detailed expressions of t^*_{as} for four different situations are presented in the following theorem, and their proofs are presented in the Appendix.

Theorem 1: Let the degradation models of the resistors under stress S_k, $k=0$, 1, be as defined in (1). Further, let

$$\omega = \frac{2\alpha_0\alpha_1 t_{l_0}^{\beta_0} - \sigma_1^2}{3\alpha_1^2}, \tag{8}$$

$$u = \frac{\left(\sigma_1^2 - 2\alpha_0\alpha_1 t_{l_0}^{\beta_0}\right)^3}{216\alpha_1^6} + \frac{\sigma_0^2\sigma_1^2 t_{l_0}^{\beta_0}}{8\alpha_1^4}, \tag{9}$$

and

$$\delta = \frac{\sigma_0^2\sigma_1^2 t_{l_0}^{\beta_0}}{64\alpha_1^{10}}\left(\alpha_1^2\sigma_0^2\sigma_1^2 t_{l_0}^{\beta_0} - \frac{16\left(\alpha_0\alpha_1 t_{l_0}^{\beta_0} - \frac{\sigma_1^2}{2}\right)^3}{27}\right). \tag{10}$$

Then, based on a decision rule for t_{as}^* in (3), we have the following.

(i) If $\delta > 0$, then

$$t_{as}^* = \sqrt[\beta_1]{\omega + \sqrt[3]{u+\sqrt{\delta}} + \sqrt[3]{u-\sqrt{\delta}}}. \tag{11}$$

(ii) If $\delta = 0$, and $2\alpha_0\alpha_1 t_{l_0}^{\beta_0} = \sigma_1^2$, then $t_{as}^* = \sqrt[\beta_1]{\omega}$.

(iii) If $\delta = 0$, and $2\alpha_0\alpha_1 t_{l_0}^{\beta_0} \neq \sigma_1^2$, then $t_{as}^* = \arg\min_{\varsigma_m, m=1,2} f(\varsigma_m)$, where

$$\varsigma_1 = \sqrt[\beta_1]{\frac{\alpha_0 t_{l_0}^{\beta_0}}{\alpha_1} - \frac{\sigma_1^2}{2\alpha_1^2} - \frac{9\sigma_0^2\sigma_1^2 t_{l_0}^{\beta_0}}{8\left(\alpha_0\alpha_1 t_{l_0}^{\beta_0} - \frac{\sigma_1^2}{2}\right)^2}}, \tag{12}$$

and

$$\varsigma_2 = \sqrt[\beta_1]{\frac{9\sigma_0^2\sigma_1^2 t_{l_0}^{\beta_0}}{\left(2\alpha_0\alpha_1 t_{l_0}^{\beta_0} - \sigma_1^2\right)^2}}. \tag{13}$$

(iv) If $\delta < 0$, then $t_{as}^* = \arg\min_{\upsilon_m, m=1,2,3} f(\upsilon_m)$, where

$$\upsilon_1 = \sqrt[\beta_1]{\omega + \sqrt[3]{u+\sqrt{\delta}} + \sqrt[3]{u-\sqrt{\delta}}}, \tag{14}$$

$$\upsilon_2 = \sqrt[\beta_1]{\omega + \frac{-1+\sqrt{3}i}{2}\sqrt[3]{u+\sqrt{\delta}} + \frac{-1-\sqrt{3}i}{2}\sqrt[3]{u-\sqrt{\delta}}}, \tag{15}$$

and

$$\upsilon_3 = \sqrt[\beta_1]{\omega + \frac{-1-\sqrt{3}i}{2}\sqrt[3]{u+\sqrt{\delta}} + \frac{-1+\sqrt{3}i}{2}\sqrt[3]{u-\sqrt{\delta}}}. \tag{16}$$

Furthermore, we obtain the probability of acceptance of the batch at time t_{as}^* in the following proposition.

Proposition 1: Let AU, and AL respectively be the pre-fixed upper, and lower limits for the acceptance of the batch in an accelerated-stress acceptance test. Then, the probability of acceptance of the batch at time t_{as}^* is given by

$$p = \Phi\left(\frac{\ln(AU+1) + \alpha_1\tau_1\left(t_{as}^*\right)}{\sigma_1\sqrt{\tau_1\left(t_{as}^*\right)}}\right) - \Phi\left(\frac{\ln(AL+1) + \alpha_1\tau_1\left(t_{as}^*\right)}{\sigma_1\sqrt{\tau_1\left(t_{as}^*\right)}}\right), \tag{17}$$

where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal random variable.

The proof of this result is presented in the Appendix.

B. Computation of AVar$(\hat{t}_{as}^*)$

Let $L_h(t_j|S_k)$ denote the sample path of the h-th tested unit at time t_j under stress S_k, for $1 \le h \le n_k$, and $1 \le j \le l_k$ when $k = 0$, 1. Let $W_{hjk} = \ln(L_h(t_j|S_k)+1) - \ln(L_h(t_{j-1}|S_k)+1)$, with $t_0 = 0$. Then, from (2), and the s-independent increment property of the Wiener process, W_{hjk} has a normal distribution as

$$W_{hjk} \sim N\left(-\alpha_k\left(\tau_k(t_j) - \tau_k(t_{j-1})\right), \sigma_k^2\left(\tau_k(t_j) - \tau_k(t_{j-1})\right)\right).$$

Hence, the log-likelihood function for the degradation paths of the resistors under stress S_0 and S_1 is given by

$$\begin{aligned}\ell(\boldsymbol{\theta}) = &-\frac{\ln(2\pi)}{2}\sum_{k=0}^{1} n_k l_k - \sum_{k=0}^{1} n_k l_k \ln(\sigma_k)\\ &-\frac{1}{2}\sum_{k=0}^{1}\sum_{j=1}^{l_k} n_k \ln\left(\tau_k(t_j) - \tau_k(t_{j-1})\right)\\ &-\sum_{k=0}^{1}\sum_{h=1}^{n_k}\sum_{j=1}^{l_k}\frac{\left[w_{hjk} + \alpha_k\left(\left(\tau_k(t_j) - \tau_k(t_{j-1})\right)\right)\right]^2}{2\sigma_k^2\left(\tau_k(t_j) - \tau_k(t_{j-1})\right)},\end{aligned} \tag{18}$$

where $\boldsymbol{\theta} = (\alpha_0, \beta_0, \sigma_0, \alpha_1, \beta_1, \sigma_1)$. By maximizing (18), the maximum likelihood estimator (MLE) $\hat{\boldsymbol{\theta}} = (\hat{\alpha}_0, \hat{\beta}_0, \hat{\sigma}_0, \hat{\alpha}_1, \hat{\beta}_1, \hat{\sigma}_1)$ of $\boldsymbol{\theta}$ can be obtained by solving the following likelihood equations.

$$\begin{aligned}&\frac{n_k\hat{\alpha}_k^2\hat{\tau}_k\left(t_{l_k}\right)\ln t_{l_k}}{\hat{\sigma}_k^2} + n_k\sum_{j=1}^{l_k}\frac{\hat{\tau}_k(t_j)\ln t_j - \hat{\tau}_k(t_{j-1})\ln t_{j-1}}{\hat{\tau}_k(t_j) - \hat{\tau}_k(t_{j-1})}\\ &= \frac{1}{\hat{\sigma}_k^2}\sum_{h=1}^{n_k}\sum_{j=1}^{l_k}\frac{w_{hjk}^2\left[\hat{\tau}_k(t_j)\ln t_j - \hat{\tau}_k(t_{j-1})\ln t_{j-1}\right]}{\left[\hat{\tau}_k(t_j) - \hat{\tau}_k(t_{j-1})\right]^2},\end{aligned} \tag{19}$$

$$\hat{\alpha}_k = -\frac{\sum_{h=1}^{n_k}\sum_{j=1}^{l_k} w_{ijk}}{n_k\hat{\tau}_k\left(t_{l_k}\right)}, \tag{20}$$

$$\hat{\sigma}_k^2 = \frac{1}{n_k l_k}\sum_{h=1}^{n_k}\sum_{j=1}^{l_k}\frac{\left[w_{hjk} + \hat{\alpha}_k\left(\hat{\tau}_k(t_j) - \hat{\tau}_k(t_{j-1})\right)\right]^2}{\hat{\tau}_k(t_j) - \hat{\tau}_k(t_{j-1})}, \tag{21}$$

where $\hat{\tau}_k(t) = t^{\hat{\beta}_k}$ for $k = 0$, 1. Then, by plugging $\hat{\boldsymbol{\theta}}$ into t_{as}^* in Theorem 1, and applying the best asymptotic normality (BAN) property of the MLE, the asymptotic variance of $\hat{t}_{as}^*$ is readily obtained to be

$$\text{AVar}\left(\hat{t}_{as}^*\right) = \left(\nabla t_{as}^*\right)' \mathbf{I}(\boldsymbol{\theta})^{-1}\left(\nabla t_{as}^*\right), \tag{22}$$

where (∇t_{as}^*) is the gradient of t_{as}^*, $(\nabla t_{as}^*)'$ is the transpose of (∇t_{as}^*), and $\mathbf{I}(\boldsymbol{\theta})$ is the Fisher information matrix. Relevant details are presented in the Appendix.

C. Cost Function

The total cost of conducting an accelerated-stress acceptance test, $\text{TC}(n_1, f_1, l_1)$, includes three different costs:

1) the cost of conducting an experiment $C_{op} f_1 l_1$,

2) the cost of measurements $C_{mea}n_1l_1$, and
3) the cost of tested devices $C_{it}n_1$.

We, therefore, have the total cost of conducting an accelerated-stress acceptance experiment to be

$$\mathrm{TC}(n_1, f_1, l_1) = C_{op}f_1l_1 + C_{mea}n_1l_1 + C_{it}n_1. \quad (23)$$

D. Optimization Problem

Under this setting, the required optimization problem is as follows.

$$\textbf{Minimize} \quad \mathrm{AVar}\left(\hat{t}^*_{as}|\boldsymbol{\xi}\right), \quad (24)$$

$$\textbf{subject to} \quad \mathrm{TC}(\boldsymbol{\xi}) \le C_b, \quad (25)$$

where $\boldsymbol{\xi} = (n_1, f_1, l_1)$, $n_1, f_1 \in \{1, 2, \ldots\}$, and $l_1 \in \{2, 3, \ldots\}$. Note that it can be easily checked that the determinant of the Fisher information matrix $\mathbf{I}(\boldsymbol{\theta})$ in (22) is zero whenever $l_1 = 1$, and for this reason the measurement number l_1 has to be at least two.

Due to the complexity of the objective function, an analytic expression for the solution of this optimization problem seems impossible. However, with the simplicity in the structure of the constraint, and the integer restriction on these decision variables, the optimal solution $\boldsymbol{\xi}^* = (n_1^*, f_1^*, l_1^*)$ can be easily determined by a complete enumeration method in a finite number of steps as described below in nine steps.

Step 1) Set $n_{1_{\max}} = \lfloor (C_b - 2C_{op})/(2C_{mea} + C_{it}) \rfloor$, where $\lfloor x \rfloor$ is the floor of x (the largest integer that is less than x), and $n_{1_{\max}}$ is the largest possible number for n_1 when $f_1 = 1$ and $l_1 = 2$.
Step 2) Set $n_1 = 1$.
Step 3) Set $l_{1_{\max}} = \lfloor (C_b - C_{it}n_1)/(C_{op} + C_{mea}n_1) \rfloor$, where $l_{1_{\max}}$ is the largest possible number for l_1 when n_1 is fixed, and $f_1 = 1$.
Step 4) Set $l_1 = 2$.
Step 5) Find f_1 such that $C_{op}f_1l_1 + C_{mea}n_1l_1 + C_{it}n_1 \le C_b$.
Step 6) Calculate $\mathrm{AVar}(\hat{t}^*_{as}|\boldsymbol{\xi})$ by $\boldsymbol{\xi}$.
Step 7) Set $l_1 = l_1 + 1$, and repeat Steps 5 and 6 until $l_1 = l_{1_{\max}}$.
Step 8) Set $n_1 = n_1 + 1$, and repeat Steps 3 through 7 until $n_1 = n_{1_{\max}}$.
Step 9) Among all possible choices of $\boldsymbol{\xi} = (n_1, f_1, l_1)$, for $1 \le n_1 \le n_{1_{\max}}$, choose that particular solution which has smallest $\mathrm{AVar}(\hat{t}^*_{as}|\boldsymbol{\xi})$ as optimal solution $\boldsymbol{\xi}^* = (n_1^*, f_1^*, l_1^*)$.

V. Motivating Example Revisited

In this section, we illustrate the proposed optimal accelerated-stress acceptance test with the resistor data introduced earlier in Section II. First, a pilot study under accelerated stress $S_1 = 125°\mathrm{C}$ was conducted, and the relative changes in resistance under stress S_1 are shown in Fig. 3. From (19), (20), and (21), we obtain the MLE of $\boldsymbol{\theta}$ as

$$\hat{\boldsymbol{\theta}} = (6.06 \times 10^{-4}, 0.20, 8.14 \times 10^{-4}, 3.29 \times 10^{-5}, 0.53, 9.01 \times 10^{-4}). \quad (26)$$

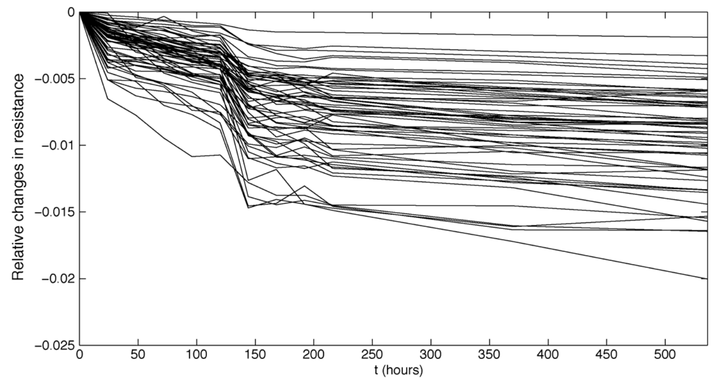

Fig. 3. Relative changes in resistance under the accelerated stress S_1.

Upon substituting $\hat{\boldsymbol{\theta}}$ into (10), we have $\hat{\delta} = 432.61 > 0$. Then, by case (i) of Theorem 1, and (22), we get the estimated optimal accelerated-stress acceptance testing time $\hat{t}^*_{as}$ to be 30.27 hours, and the corresponding estimated asymptotic variance $\widehat{\mathrm{AVar}}(\hat{t}^*_{as})$ as 54.79, which yields a 95% normal-approximate confidence interval (CI) for t^*_{as} as

$$\hat{t}^*_{as} \pm 1.96\sqrt{\widehat{\mathrm{AVar}}\left(\hat{t}^*_{as}\right)} = [15.76, 44.78].$$

It is clear that the acceptance testing time under the accelerated test is considerably more efficient than the traditional acceptance testing time (with corresponding efficiency being $(1000/30.27) \times 100\% = 3304\%$). Furthermore, from Proposition 1 with commonly used values of $AU = 0.01$ and $AL = -0.01$, the estimated probability of acceptance of the batch can be obtained as

$$\Phi\left(\frac{\ln(0.01+1) + 3.29 \times 10^{-5} \times 30.27^{0.53}}{9.01 \times 10^{-4} \times \sqrt{30.27^{0.53}}}\right) - \Phi\left(\frac{\ln(-0.01+1) + 3.29 \times 10^{-5} \times 30.27^{0.53}}{9.01 \times 10^{-4} \times \sqrt{30.27^{0.53}}}\right) = 0.9998.$$

From (22), it is known that $\mathrm{AVar}(\hat{t}^*_{as}|\boldsymbol{\xi})$ is a function of $\boldsymbol{\theta}$ when the test plan $\boldsymbol{\xi}$ is given. Hence, for determining an optimal accelerated-stress acceptance test, we need information on the parameter vector $\boldsymbol{\theta}$. In what follows, we take the MLE of $\boldsymbol{\theta}$ in (26) as the true model parameter, and then describe the construction of the optimal design for the resistor data.

Suppose the cost factors C_{it}, C_{op}, and C_{mea} are

$$C_{it} = \$53/\mathrm{coupon},$$
$$C_{mea} = \$0.3/\mathrm{measurement},$$
$$C_{op} = \$11/\mathrm{unit\ time};$$

and the unit time t_{u_1} is 24 hours (i.e., one day). Then, under various choices of budget specification C_b, the optimal test plans for the accelerated-stress acceptance test can be obtained by using the algorithm described in Section IV-D, and the results so obtained are presented in Table I.

1) **Optimal Test Plan**

For example, when the budget $C_b = 2000$, the optimal test plan turns out to be $(n_1^*, f_1^*, l_1^*) = (30, 1, 20)$. In other words, the optimal sample size is 30, the optimal measurement frequency is $1 \times 24 = 24$ hours, and the optimal measurement number is 20. Thus, the total test

TABLE I
OPTIMAL TEST PLANS UNDER VARIOUS CHOICES OF BUDGET SPECIFICATION C_b

C_b	n_1^*	f_1^*	l_1^*	AVar$(\hat{t}_{as}^*\|\boldsymbol{\xi}^*)$	Total test cost
2000	30	1	20	82.03	1990.00
2250	34	1	21	74.53	2247.20
2500	35	1	30	68.71	2500.00

time for this accelerated-stress acceptance experiment is $1 \times 20 \times 24 = 480$ hours. Under such a test plan, the total cost is 1990.00, and the corresponding asymptotic variance of the estimated optimal accelerated-stress acceptance testing time is 82.03.

2) **Sensitivity Analysis**
In practice, the estimated parameter $\hat{\boldsymbol{\theta}} = (\hat{\alpha}_0, \hat{\beta}_0, \hat{\sigma}_0, \hat{\alpha}_1, \hat{\beta}_1, \hat{\sigma}_1)$ would depart from the true parameter $\boldsymbol{\theta} = (\alpha_0, \beta_0, \sigma_0, \alpha_1, \beta_1, \sigma_1)$. Suppose δ_m $(m = 1, \cdots, 6)$ denote the errors in the specification of the parameters α_0, β_0, σ_0, α_1, β_1, and σ_1, respectively. We now use the $L_r(g^e)$ design matrix for sensitivity analysis (which can be easily generated by JMP statistical software) to examine the effects of misspecification of these parameters on the optimal test plan. Here, r is the number of runs (used in the design), g is the number of levels of each factor (or parameter), and e is the number of factors (or parameters). Under the same cost configuration $(C_{it}, C_{mea}, C_{op}, C_b) = (53, 0.3, 11, 2000)$, the optimal test plans for various choices of $((1+\delta_1)\alpha_0, (1+\delta_2)\beta_0, (1+\delta_3)\sigma_0, (1+\delta_4)\alpha_1, (1+\delta_5)\beta_1, (1+\delta_6)\sigma_1)$ are presented in Table II in the form of an $L_{13}(3^6)$ design matrix. From these results, we observe that the optimal test plan (n_1^*, f_1^*, l_1^*) is quite robust to moderate departures from the assumed value of $\boldsymbol{\theta}$. Note that the optimal value f_1^* is quite insensitive to the model parameter $\boldsymbol{\theta}$. However, the optimal sample size n_1^* and the measurement number l_1^* are somewhat sensitive. Hence, for designing a good accelerated-stress acceptance plan, we need a precise parameter specification.

3) **Simulation Study**
The above results are based on the asymptotic normality of the MLE. In this section, a Monte Carlo simulation study is performed to show that the asymptotic results can be closely approximated by those obtained from simulations. Corresponding to the resistor data presented earlier, the true parameter settings of the degradation model of the resistors under stress levels S_0 and S_1 were assumed as in (26), and 1000 sets of degradation data were generated. Table III presents the true and simulated values of the parameters, and the optimal accelerated-stress acceptance testing time t_{as}^* in the model. The corresponding asymptotic and simulated variances of the estimated parameters and the optimal accelerated-stress acceptance testing time are also given inside the parentheses. We observe that all the simulated values and the variances of parameters are quite close to the corresponding true values and their asymptotic variances.

TABLE II
OPTIMAL TEST PLANS UNDER VARIOUS CHOICES OF THE PARAMETERS $((1+\delta_1)\alpha_0, (1+\delta_2)\beta_0, (1+\delta_3)\sigma_0, (1+\delta_4)\alpha_1, (1+\delta_5)\beta_1, (1+\delta_6)\sigma_1)$

δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	n_1^*	f_1^*	l_1^*
+5%	+5%	−5%	0	−5%	0	33	1	12
−5%	−5%	+5%	0	−5%	−5%	29	1	23
−5%	0	0	0	0	0	29	1	23
−5%	+5%	+5%	+5%	0	+5%	28	1	26
+5%	−5%	0	+5%	−5%	+5%	29	1	23
0	0	−5%	+5%	−5%	−5%	29	1	23
+5%	0	+5%	0	+5%	+5%	30	1	20
0	−5%	−5%	0	0	+5%	29	1	23
−5%	0	−5%	−5%	−5%	+5%	29	1	23
−5%	−5%	−5%	+5%	+5%	0	26	1	33
+5%	−5%	−5%	−5%	0	−5%	33	1	12
0	+5%	0	−5%	+5%	−5%	33	1	12
0	−5%	+5%	−5%	−5%	0	33	1	12
0	0	0	0	0	0	30	1	20

TABLE III
RESULTS OF A MONTE CARLO SIMULATION STUDY FOR THE ESTIMATES OF THE MODEL PARAMETERS AND t_{as}^*

Parameter	True value (asymptotic variance)	Simulated value (simulated variance)
α_0	6.06×10^{-4} (7.00×10^{-9})	6.02×10^{-4} (6.86×10^{-9})
β_0	0.20 (2.39×10^{-4})	0.19 (2.34×10^{-4})
σ_0	8.14×10^{-4} (4.17×10^{-9})	8.09×10^{-4} (3.91×10^{-9})
α_1	3.29×10^{-5} (4.19×10^{-9})	3.29×10^{-5} (4.18×10^{-9})
β_1	0.53 (8.68×10^{-4})	0.54 (8.87×10^{-4})
σ_1	9.01×10^{-4} (7.92×10^{-9})	8.96×10^{-4} (7.86×10^{-9})
t_{as}^*	30.27 (54.79)	31.23 (57.28)

VI. CONCLUDING REMARKS

An acceptance test is an extremely important stage to ensure products meet the customer's requirements. Note, however, that traditional acceptance tests based on time-to-failure data are no longer practical for highly reliable products. For this situation, motivated by resistor data, an accelerated-stress acceptance test based on a Wiener degradation model has been proposed here to shorten the acceptance testing time. As illustrated in Section V, the acceptance testing time under the accelerated test is considerably better than the traditional acceptance testing time.

Next, by minimizing the asymptotic variance of the estimated optimal accelerated-stress acceptance testing time subject to the total experimental cost not exceeding a pre-specified budget, we have determined the optimal test plan, including the total sample size, measurement frequency, and the number of measurements, for an accelerated-stress acceptance test. The sensitivity analysis carried out shows that the optimal test plan is quite robust. A simulation study reveals that all the estimated parameter values and the variances of the estimates are quite close to the corresponding true values and their asymptotic variances.

Some possible extensions of this work can be made in the following directions.

(i) The degradation model for the resistor data in (2) can be extended to $m(>0)$ higher levels of stress (temperature), $(S_0) < S_1 < S_2 < \cdots < S_m$. This condition is also called an accelerated-stress acceptance test with m levels of constant stress. In such a case, the (Arrhenius) relationship between accelerating variable (S_k) and decay rate (α_k) of the resistors can be incorporated into the degradation model. However, the simple decision rule in (3) is no longer suitable for handling the higher-level problem. So, more appropriate decision rules for optimal accelerated-stress acceptance testing times $t_{as}^{*(k)}$ under different stresses S_k, $k = 1, \ldots, m$, need to be suggested in advance.

(ii) There may exist other extra sources of variability in the resistors. A random-effect degradation model that describes such product-to-product variability might be able to make the proposed model applicable to a much larger range of real-life data sets (Lu & Meeker, [9]).

(iii) Sometimes, degradation paths of the products may be compounded and contaminated by measurement errors. Under such a situation, it may be more appropriate to fit a general Wiener degradation model with measurement errors (Whitmore, [31]) for the resistor data. Another area would be to extend the proposed method based on a gamma degradation process to handle fatigue data with a monotone-increasing pattern (Lawless & Crowder, [7]).

(iv) For some poor-quality resistors in the batch, their relative changes in resistance would be decayed or diffused much faster than would the good-quality ones. For such poor-quality batches, the probability of acceptance is the consumer's risk, and the probability of rejection of a good-quality batch is the producer's risk. It is important to balance these two risks by incorporating their respective costs in the proposed decision model. Methods like those described by Tsai *et al.* [23] could be developed in this context.

These problems are of great interest, and we hope to consider these issues for our future research.

APPENDIX

Proof of Theorem 1:

Proof: Taking derivative of (7) with respect to t, we have

$$\left(2\alpha_1^2\right)\left(\tau_1\left(t_{as}^*\right)\right)^2 + \left(\sigma_1^2 - 2\alpha_0\alpha_1 t_{l_0}^{\beta_0}\right)\tau_1\left(t_{as}^*\right) = \left(\sigma_0\sigma_1\sqrt{t_{l_0}^{\beta_0}}\right)\sqrt{\tau_1\left(t_{as}^*\right)},$$

which, upon dividing by $\sqrt{\tau_1(t_{as}^*)}$ and taking squares on both sides, yields a cubic equation as

$$a\left(\tau_1\left(t_{as}^*\right)\right)^3 + b\left(\tau_1\left(t_{as}^*\right)\right)^2 + c\tau_1\left(t_{as}^*\right) + d = 0, \tag{27}$$

where

$$a = 4\alpha_1^4, \tag{28}$$

$$b = -8\alpha_1^2\left(\alpha_0\alpha_1 t_{l_0}^{\beta_0} - \frac{\sigma_1^2}{2}\right), \tag{29}$$

$$c = \left(2\alpha_0\alpha_1 t_{l_0}^{\beta_0} - \sigma_1^2\right)^2 \tag{30}$$

and

$$d = -\sigma_0^2\sigma_1^2 t_{l_0}^{\beta_0}. \tag{31}$$

It is well known that by the intermediate value theorem, every cubic equation with real coefficients has at least one real solution, which can be determined through the discriminant given by

$$\delta = \left(\frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3. \tag{32}$$

Then, upon substituting the expressions in (28)–(31) into (32), we obtain the explicit expression of δ in (10). Based on the values of δ and $b^2 - 3ac$, four types of roots for $\tau_1(t_{as}^*)$ can be obtained as detailed below.

(i) if $\delta > 0$, then the cubic equation in (27) has only one real root as

$$\tau_1\left(t_{as}^*\right) = \omega + \sqrt[3]{u + \sqrt{\delta}} + \sqrt[3]{u - \sqrt{\delta}},$$

where

$$\omega = -\frac{b}{3a} \tag{33}$$

and

$$u = \frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a}. \tag{34}$$

Thus, upon substituting the expressions (28) through (31) into (33) and (34), we obtain the corresponding optimal accelerated-stress acceptance testing time, in this case as

$$t_{as}^* = \sqrt[\beta_1]{\omega + \sqrt[3]{u + \sqrt{\delta}} + \sqrt[3]{u - \sqrt{\delta}}},$$

where ω, and u are as expressed in (8), and (9), respectively.

(ii) If $\delta = 0$, and $b^2 - 3ac = 0$, then the cubic equation in (27) has three equal real roots as

$$\tau_1\left(t_{as}^*\right) = -\frac{b}{3a}. \tag{35}$$

Upon substituting the expressions in (28) through (30) into $b^2 - 3ac = 0$, we obtain $2\alpha_0\alpha_1 t_{l_0}^{\beta_0} = \sigma_1^2$. Thus, the optimal accelerated-stress acceptance testing time in such a case is

$$t_{as}^* = \sqrt[\beta_1]{\omega},$$

where ω is as given in (8).

(iii) if $\delta = 0$, and $b^2 - 3ac \neq 0$, then the cubic equation in (27) has two equal real roots

$$\tau_1(\varsigma_1) = \frac{bc - 9ad}{2(3ac - b^2)}, \tag{36}$$

and one simple root

$$\tau_1(\varsigma_2) = \frac{9a^2d - 4abc + b^3}{a(3ac - b^2)}. \tag{37}$$

Thus, upon substituting the expressions in (28)through (31) into (36) and (37), we obtain the optimal accelerated-stress acceptance testing time in this case as

$$t_{as}^* = \underset{\varsigma_m, m=1,2}{\arg\min}\ f(\varsigma_m),$$

where ς_1 and ς_2 are as given in (12) and (13).

(iv) if $\delta < 0$, then the cubic equation in (27) has three distinct real roots as

$$\tau_1(\upsilon_1) = \omega + \sqrt[3]{u + \sqrt{\delta}} + \sqrt[3]{u - \sqrt{\delta}},$$
$$\tau_1(\upsilon_2) = \omega + \frac{-1 + \sqrt{3}i}{2}\sqrt[3]{u + \sqrt{\delta}} + \frac{-1 - \sqrt{3}i}{2}\sqrt[3]{u - \sqrt{\delta}},$$

and

$$\tau_1(\upsilon_3) = \omega + \frac{-1 - \sqrt{3}i}{2}\sqrt[3]{u + \sqrt{\delta}} + \frac{-1 + \sqrt{3}i}{2}\sqrt[3]{u - \sqrt{\delta}},$$

where ω, and u are as defined in (33), and (34), respectively.

In this case, we obtain the optimal accelerated-stress acceptance testing time as

$$t_{as}^* = \underset{\upsilon_m, m=1,2,3}{\arg\min}\ f(\upsilon_m),$$

where υ_1, υ_2, and υ_3 are as given in (14), (15), and (16), respectively.

□

Detailed Expressions of $(\nabla t_{as}^*)'$ *and* $\mathbf{I}(\boldsymbol{\theta})$ *in (22):*

Proof: We have

$$(\nabla t_{as}^*)' = \left(\frac{\partial t_{as}^*}{\partial \alpha_0}, \frac{\partial t_{as}^*}{\partial \beta_0}, \frac{\partial t_{as}^*}{\partial \sigma_0}, \frac{\partial t_{as}^*}{\partial \alpha_1}, \frac{\partial t_{as}^*}{\partial \beta_1}, \frac{\partial t_{as}^*}{\partial \sigma_1}\right).$$

From the data analysis in Section V, the discriminant δ in (10) is greater than 0. Then, by part (i) of Theorem 1, and logarithmic differentiation, the elements of the gradient $(\nabla t_{as}^*)'$ in such a case are given by

$$\begin{aligned}\frac{\partial t_{as}^*}{\partial \alpha_k} &= \frac{\tau_1\left(t_{as}^*\right)^{1/\beta_1 - 1}}{\beta_1} \\ &\times \left[\frac{\partial \omega}{\partial \alpha_k} + \frac{(u+\sqrt{\delta})^{-2/3}}{3}\left(\frac{\partial u}{\partial \alpha_k} + \frac{1}{2\sqrt{\delta}}\frac{\partial \delta}{\partial \alpha_k}\right)\right. \\ &\left. + \frac{(u-\sqrt{\delta})^{-2/3}}{3}\left(\frac{\partial u}{\partial \alpha_k} - \frac{1}{2\sqrt{\delta}}\frac{\partial \delta}{\partial \alpha_k}\right)\right],\end{aligned}$$

$$\begin{aligned}\frac{\partial t_{as}^*}{\partial \beta_0} &= \frac{\tau_1\left(t_{as}^*\right)^{1/\beta_1 - 1}}{\beta_1} \\ &\times \left[\frac{\partial \omega}{\partial \beta_0} + \frac{(u+\sqrt{\delta})^{-2/3}}{3}\left(\frac{\partial u}{\partial \beta_0} + \frac{1}{2\sqrt{\delta}}\frac{\partial \delta}{\partial \beta_0}\right)\right. \\ &\left. + \frac{(u-\sqrt{\delta})^{-2/3}}{3}\left(\frac{\partial u}{\partial \beta_0} - \frac{1}{2\sqrt{\delta}}\frac{\partial \delta}{\partial \beta_0}\right)\right],\\
\frac{\partial t_{as}^*}{\partial \beta_1} &= -\frac{1}{\beta_1^2}\tau_1\left(t_{as}^*\right)^{\frac{1}{\beta_1}}\ln \tau_1\left(t_{as}^*\right),\\
\frac{\partial t_{as}^*}{\partial \sigma_k} &= \frac{\tau_1\left(t_{as}^*\right)^{1/\beta_1 - 1}}{\beta_1} \\ &\times \left[\frac{\partial \omega}{\partial \sigma_k} + \frac{(u+\sqrt{\delta})^{-2/3}}{3}\left(\frac{\partial u}{\partial \sigma_k} + \frac{1}{2\sqrt{\delta}}\frac{\partial \delta}{\partial \sigma_k}\right)\right. \\ &\left. + \frac{(u-\sqrt{\delta})^{-2/3}}{3}\left(\frac{\partial u}{\partial \sigma_k} - \frac{1}{2\sqrt{\delta}}\frac{\partial \delta}{\partial \sigma_k}\right)\right],\end{aligned}$$

for $k = 0, 1$, where we see the equation at the bottom of the next page.

Finally, we obtain the Fisher information matrix $\mathbf{I}(\boldsymbol{\theta})$ as

$$\mathbf{I}(\boldsymbol{\theta}) = \begin{bmatrix}\mathbf{I}_0(\boldsymbol{\theta}) & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_1(\boldsymbol{\theta})\end{bmatrix},$$

where $\mathbf{0}$ is a 3×3 matrix of zeros; and, for $k = 0, 1$,

$$\mathbf{I}_k(\boldsymbol{\theta}) = \begin{bmatrix}\mathrm{E}\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \alpha_k^2}\right) & \mathrm{E}\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \alpha_k \partial \beta_k}\right) & \mathrm{E}\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \alpha_k \partial \sigma_k}\right) \\ \mathrm{E}\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \beta_k \partial \alpha_k}\right) & \mathrm{E}\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \beta_k^2}\right) & \mathrm{E}\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \beta_k \partial \sigma_k}\right) \\ \mathrm{E}\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \sigma_k \partial \alpha_k}\right) & \mathrm{E}\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \sigma_k \partial \beta_k}\right) & \mathrm{E}\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \sigma_k^2}\right)\end{bmatrix},$$

where

$$\begin{aligned}\mathrm{E}\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \alpha_k^2}\right) &= \frac{n_k \tau_k(t_{l_k})}{\sigma_k^2},\\
\mathrm{E}\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \alpha_k \partial \beta_k}\right) &= \mathrm{E}\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \beta_k \partial \alpha_k}\right) = \frac{n_k \alpha_k \tau_k(t_{l_k}) \ln t_{l_k}}{\sigma_k^2},\\
\mathrm{E}\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \alpha_k \partial \sigma_k}\right) &= \mathrm{E}\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \sigma_k \partial \alpha_k}\right) = 0,\\
\mathrm{E}\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \beta_k^2}\right) &= n_k \sum_{j=1}^{l_k} \\ &\times\left[\left(\frac{1}{2} + \left(\frac{\alpha_k}{\sigma_k}\right)^2 (\tau_k(t_j) - \tau_k(t_{j-1}))\right)\right. \\ &\left.\times\left(\frac{\tau_k(t_j)\ln t_j - \tau_k(t_{j-1})\ln t_{j-1}}{\tau_k(t_j) - \tau_k(t_{j-1})}\right)^2\right],\\
\mathrm{E}\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \beta_k \partial \sigma_k}\right) &= \mathrm{E}\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \sigma_k \partial \beta_k}\right) \\ &= \frac{n_k}{\sigma_k}\sum_{j=1}^{l_k}\left(\frac{\tau_k(t_j)\ln t_j - \tau_k(t_{j-1})\ln t_{j-1}}{\tau_k(t_j) - \tau_k(t_{j-1})}\right),\\
\mathrm{E}\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \sigma_k^2}\right) &= \frac{2 n_k l_k}{\sigma_k^2}.\end{aligned}$$

□

Proof of Proposition 1:

Proof: From (1) and (2), see that the probability of acceptance of the batch at fixed time t under the accelerated-stress acceptance experiment is given by

$$P\left(AL \le L(t|S_1) \le AU\right) = P\left(\ln(AL+1) \le W_1(t) \le \ln(AU+1)\right).$$

Then, from (5) and (6), we have

$$\frac{W_1(t) + \alpha_1 \tau_1(t)}{\sigma_1\sqrt{\tau_1(t)}} \sim N(0,1).$$

Thus, the required result follows. □

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$$
\begin{aligned}
\frac{\partial \omega}{\partial \alpha_0} &= \frac{2t_{l_0}^{\beta_0}}{3\alpha_1},\\
\frac{\partial u}{\partial \alpha_0} &= -\frac{t_{l_0}^{\beta_0}\left(\sigma_1^2 - 2\alpha_0\alpha_1 t_{l_0}^{\beta_0}\right)^2}{36\alpha_1^5},\\
\frac{\partial \delta}{\partial \alpha_0} &= -\frac{\sigma_0^2\sigma_1^2 t_{l_0}^{2\beta_0}\left(\sigma_1^2 - 2\alpha_0\alpha_1 t_{l_0}^{\beta_0}\right)^2}{144\alpha_1^9},\\
\frac{\partial \omega}{\partial \alpha_1} &= \frac{2\left(\sigma_1^2 - \alpha_0\alpha_1 t_{l_0}^{\beta_0}\right)}{3\alpha_1^3},\\
\frac{\partial u}{\partial \alpha_1} &= \frac{\left(\alpha_0\alpha_1 t_{l_0}^{\beta_0} - \sigma_1^2\right)\left(2\alpha_0\alpha_1 t_{l_0}^{\beta_0} - \sigma_1^2\right)^2 - 18\alpha_1^2\sigma_0^2\sigma_1^2 t_{l_0}^{\beta_0}}{36\alpha_1^7},\\
\frac{\partial \delta}{\partial \alpha_1} &= \frac{\sigma_0^2\sigma_1^2 t_{l_0}^{\beta_0}\left[\left(7\alpha_0\alpha_1 t_{l_0}^{\beta_0} - 5\sigma_1^2\right)\left(2\alpha_0\alpha_1 t_{l_0}^{\beta_0} - \sigma_1^2\right)^2 - 54\alpha_1^2\sigma_0^2\sigma_1^2 t_{l_0}^{\beta_0}\right]}{432\alpha_1^{11}},\\
\frac{\partial \omega}{\partial \beta_0} &= \frac{2\alpha_0 t_{l_0}^{\beta_0}\ln t_{l_0}}{3\alpha_1},\\
\frac{\partial u}{\partial \beta_0} &= \frac{\sigma_0^2\sigma_1^2 t_{l_0}^{\beta_0}\ln t_{l_0}}{8\alpha_1^4} - \frac{\alpha_0 t_{l_0}^{\beta_0}\ln t_{l_0}\left(\sigma_1^2 - 2\alpha_0\alpha_1 t_{l_0}^{\beta_0}\right)^2}{36\alpha_1^5},\\
\frac{\partial \delta}{\partial \beta_0} &= \frac{\sigma_0^2\sigma_1^2 t_{l_0}^{\beta_0}\ln t_{l_0}\left[27\alpha_1^2\sigma_0^2\sigma_1^2 t_{l_0}^{\beta_0} + \left(\sigma_1^2 - 8\alpha_0\alpha_1 t_{l_0}^{\beta_0}\right)\left(\sigma_1^2 - 2\alpha_0\alpha_1 t_{l_0}^{\beta_0}\right)^2\right]}{864\alpha_1^{10}},\\
\frac{\partial \omega}{\partial \sigma_0} &= 0,\\
\frac{\partial u}{\partial \sigma_0} &= \frac{\sigma_0\sigma_1^2 t_{l_0}^{\beta_0}}{4\alpha_1^4},\\
\frac{\partial \delta}{\partial \sigma_0} &= \frac{\sigma_0\sigma_1^2 t_{l_0}^{\beta_0}\left[27\alpha_1^2\sigma_0^2\sigma_1^2 t_{l_0}^{\beta_0} + \left(\sigma_1^2 - 2\alpha_0\alpha_1 t_{l_0}^{\beta_0}\right)^3\right]}{432\alpha_1^{10}},\\
\frac{\partial \omega}{\partial \sigma_1} &= -\frac{2\sigma_1}{3\alpha_1^2},\\
\frac{\partial u}{\partial \sigma_1} &= \frac{\sigma_1\left[9\alpha_1^2\sigma_0^2 t_{l_0}^{\beta_0} + \left(\sigma_1^2 - 2\alpha_0\alpha_1 t_{l_0}^{\beta_0}\right)^2\right]}{36\alpha_1^6},\\
\frac{\partial \delta}{\partial \sigma_1} &= \frac{\sigma_0^2\sigma_1 t_{l_0}^{\beta_0}\left[27\alpha_1^2\sigma_0^2\sigma_1^2 t_{l_0}^{\beta_0} + 2\left(2\sigma_1^2 - \alpha_0\alpha_1 t_{l_0}^{\beta_0}\right)\left(\sigma_1^2 - 2\alpha_0\alpha_1 t_{l_0}^{\beta_0}\right)^2\right]}{432\alpha_1^{10}}
\end{aligned}
$$

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